

Developing Understanding in Mathematics via Problem Solving

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EARLY in this decade the theme of school mathematics shifted from "back to the basics" to "problem solving." In fact, in recent years problem solving has been the most written-about and talked-about part of the mathematics curriculum and at the same time the least understood. Now that there has been nearly a decade of attempts to make problem solving "the focus of school mathematics" (NCTM, 1980, p. 1), we need to assess the results of these efforts. This article addresses the role of problem solving in elementary school mathematics in the hope of adding some much-needed clarity to the discussion. Our main point is that the most important role for problem solving is to develop students' understanding of mathematics.

APPROACHES TO PROBLEM-SOLVING INSTRUCTION

In the main, the discussions about problem solving and the efforts to develop curricula and materials for students and teachers have been worthwhile and helpful. Today the notion that problem solving should play a prominent role in the curriculum has widespread acceptance. During the past decade quite a large number of problem-solving resources have been developed for classroom use in the form of collections of problems, lists of strategies to be taught, suggestions for activities, and guidelines for evaluating problem-solving performance. Much of this material has been very useful in helping teachers make problem solving a focus of their instruction. However, it has not provided the sort of coherence and clear direction that is

needed, primarily because to date little agreement has been reached on *how* this goal is to be achieved. Undoubtedly there are several reasons for this state of affairs, but the confusion probably stems from the vast differences among individuals' and groups' conceptions of what it means to make problem solving the focus of school mathematics. One of the best ways of coming to grips with these differences is to distinguish among three approaches to problem-solving instruction: (1) teaching *about* problem solving, (2) teaching *for* problem solving, and (3) teaching *via* problem solving. An explicit statement of this distinction appeared in a paper written more than a decade ago by Hatfield (1978), but we suspect that others may have espoused a similar point of view as well. Let us explain what each of these three approaches entails.

Teaching about Problem Solving

The teacher who teaches *about* problem solving highlights Pólya's (1957) model of problem solving (or some minor variation of it). Briefly, this model describes a set of four interdependent phases in the process of solving mathematics problems: ^{realize} understanding the problem, ^{the solution} devising a plan, ^{often as well} carrying out the plan, and looking back. Students are explicitly taught the phases that, according to Pólya, expert problem solvers use when solving mathematics problems, and they are encouraged to become aware of their own progression through these phases when they themselves solve problems. Additionally, they are taught a number of "heuristics," or "strategies," from which they can choose or which they should use in devising and carrying out their problem-solving plans. Some of the strategies typically taught include looking for patterns, solving a simpler problem, and working backward. At its best, teaching about problem solving also includes experiences with actually solving problems, but it always involves a great deal of explicit discussion of, and teaching about, how problems are solved.

Teaching for Problem Solving

In teaching *for* problem solving, the teacher concentrates on ways in which the mathematics being taught can be applied in the solution of both routine and nonroutine problems. Although the acquisition of mathematical knowledge is of primary importance, the essential purpose for learning mathematics is to be able to use it. Consequently, students are given many instances of the mathematical concepts and structures they are studying and many opportunities to apply that mathematics in solving problems. Further, the teacher who teaches *for* problem solving is very concerned about students' ability to transfer what they have learned from one problem context to others. A strong adherent ^{of this approach} of this approach might argue that the sole reason for learning mathematics is to be able to use the knowledge gained to solve problems.

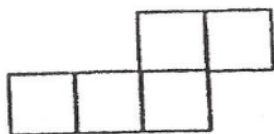
Teaching via Problem Solving

In teaching *via* problem solving, problems are valued not only as a purpose for learning mathematics but also as a primary means of doing so. The teaching of a mathematical topic begins with a problem situation that embodies key aspects of the topic, and mathematical techniques are developed as reasonable responses to reasonable problems. A goal of learning mathematics is to transform certain nonroutine problems into routine ones. The learning of mathematics in this way can be viewed as a movement from the concrete (a real-world problem that serves as an instance of the mathematical concept or technique) to the abstract (a symbolic representation of a class of problems and techniques for operating with these symbols).

An example from the Middle Grades Mathematics Project can serve to illustrate teaching via problem solving (Shroyer and Fitzgerald 1986). A fifth-grade teacher who has decided to introduce the concepts of area and perimeter gives each student a set of twenty-four one-inch-square tiles that are to be regarded as small tables. The students are challenged to determine the number of small tables (tiles) needed to make banquet tables of different sizes (area) and the number of people who can be seated at these banquet tables (perimeter). The students are told that one small table can seat four people, one on each side, and that the banquet tables made from the small tables are usually rectangular. The real-world situation (forming banquet tables and seating people around them) serves as a context in which students explore area and perimeter and the relationships between them. At first no formulas are used or developed; they will come in a later activity. Examples of the challenges presented by the teacher include the following (Shroyer and Fitzgerald 1986):

Example A: Use your tiles to make different arrangements that will seat twenty people.

Example B: Add squares to the following arrangement so that the perimeter is 18. What is the new area?



Some Observations about the Three Approaches

Although in theory these three conceptions of teaching problem solving in mathematics can be isolated, in practice they overlap and occur in various combinations and sequences. Thus, it is probably counterproductive to argue in favor of one or more of these types of teaching or against the others.

Nevertheless, if curriculum developers, textbook writers, or classroom teachers intend to make problem solving the "focus of instruction," they need to be aware of the limitations inherent in exclusive adherence to either of the first two types of problem-solving instruction. One such limitation stems from the fact that problem solving is not a topic of mathematics, and it should not be regarded as such. If teaching *about* problem solving is the focus, the danger is that "problem solving" will be regarded as a strand to be added to the curriculum. Instead of problem solving serving as a context in which mathematics is learned and applied, it may become just another topic, taught in isolation from the content and relationships of mathematics.

A different ^{definition}shortcoming arises from teaching *for* problem solving. When this approach is interpreted narrowly, problem solving is viewed as an activity students engage in only *after* the introduction of a new concept or following work on a computational skill or algorithm. The purpose is to give students an opportunity to "apply" recently learned concepts and skills to the solution of real-world problems. Often these problems appear under a heading such as "Using Division to Solve Problems," and a solution of a sample story problem is given as a model for solving other, very similar problems. Often, solutions to these problems can be obtained simply by following the pattern established in the sample, and when students encounter problems that do not follow the sample, they often feel at a loss. It has been our experience (which is supported by several studies) that when taught in this way, students often simply pick out the numbers in each problem and apply the given operation(s) to them without regard for the problem's context; as often as not, they obtain the correct answers. In our view this practice is certainly not problem solving. Indeed, it does not even require mathematical thinking. Furthermore, a side effect is that students come to believe that all mathematics problems can be solved quickly and relatively effortlessly without any need to understand how the mathematics they are using relates to real situations. Unfortunately, this approach to problem-solving instruction has been quite common in textbooks.

Unlike the other two approaches, teaching *via* problem solving is a conception that has not been adopted either implicitly or explicitly by many teachers, textbook writers, and curriculum developers, but it is an approach to the teaching of mathematics that deserves to be considered, developed, tried, and evaluated. Indeed, teaching *via* problem solving is the approach that is most consistent with the recommendations of NCTM's Standards Commission that (1) mathematics concepts and skills be learned in the context of solving problems; (2) the development of higher-level thinking processes be fostered through problem-solving experiences; and (3) mathematics instruction take place in an inquiry-oriented, problem-solving atmosphere (NCTM 1987).

TWO MODELS OF THE PROCESS OF SOLVING MATHEMATICS PROBLEMS

Problem solving has sometimes been conceptualized in a simplistic way by a model like that in figure 3.1. This model has two levels, or "worlds": the everyday world of things, problems, and applications of mathematics; and the idealized, abstract world of mathematical symbols, operations, and techniques. In this model the problem-solving process has three steps: Beginning with a problem posed in terms of the everyday physical reality, the problem solver first translates (arrow A) the problem into abstract mathematical terms, then operates (arrow B) on the mathematical representation to come to a mathematical solution of the problem, which is then translated back (arrow C) into the terms of the original problem.

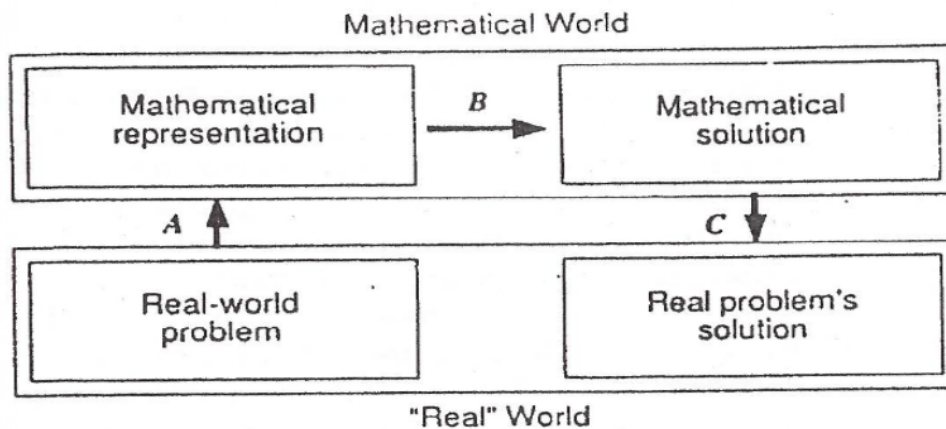


Fig. 3.1. A simplistic model of the process of solving mathematics problems

According to this model, mathematics can be, and often is, learned separately from its applications. In teaching *for* problem solving, instructors are very concerned to develop students' abilities to translate real-world problems into mathematical representations, and vice versa. But they tend to deal with problems and applications of mathematics only *after* those mathematical concepts and skills have been introduced, developed, and practiced. The difficulty with this model is that it applies to routine problems better than to nonroutine ones. Problems classified as "translation problems" (Charles and Lester 1982) are solved exactly as the model indicates, but for more challenging problems, like those categorized by Charles and Lester as "process problems," the problem solver has no single already-learned mathematical operation that will solve the problem. As well as translation and interpretation, these nonroutine problems also demand more complex processes, such as planning, selecting a strategy, identifying subgoals, conjecturing, and verifying that a solution has been found. For nonroutine problems, a different type of model is required.

Figure 3.2 shows a modification of the problem-solving model for translation problems that can be used to illustrate thinking processes when nonroutine problems are involved and when teaching *via* problem solving is adopted. This model also contains two levels that represent the everyday world of problems and the abstract world of mathematical symbols and operations. In this model, however, the mathematical processes in the upper level are "under construction" (i.e., being learned, as opposed to already learned), and its most important features are the relationships between the steps in the mathematical process (in the upper level) and the actions on particular elements in the problems (in the lower level).

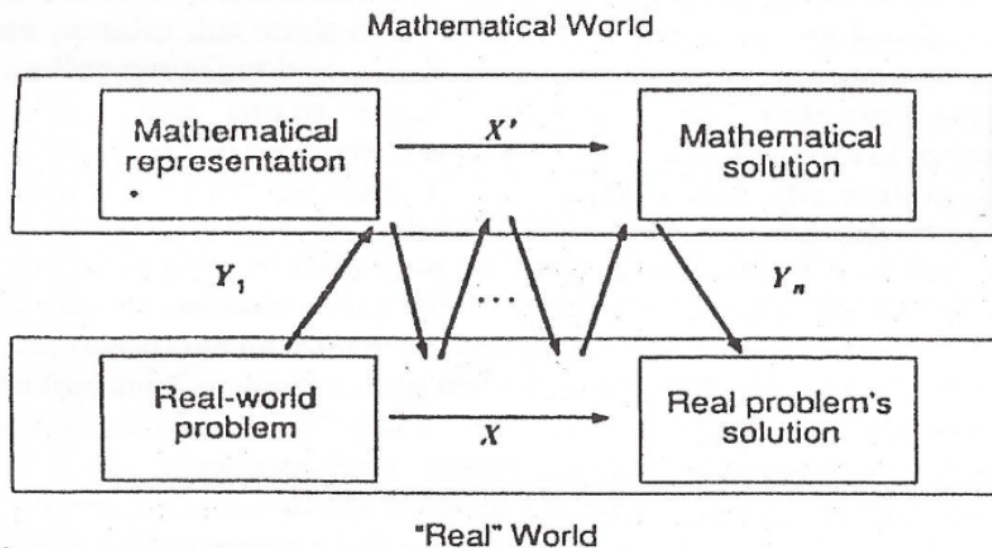


Fig. 3.2. A model of the process of solving process problems

In the figure, some of the Y arrows point upward to indicate that the problem solver is learning to make abstract written records of the actions that are understood in a concrete setting. These arrows pointing upward represent the processes of abstraction and generalization. Some of the arrows point downward to show that the problem solver is able to explain a mathematical process by referring to the real-world actions that the mathematical symbols represent. Arrows pointing downward might also suggest that a problem solver who had forgotten the details of a mathematical procedure would be able to reconstruct that process by imagining the corresponding concrete steps in the world in which the problem was posed. The collection of Y arrows illustrate the correspondence between the process of solving the problem in concrete terms (labeled X) and the parallel, abstract mathematical process (labeled X'). The Y arrows also show that the problem solver typically moves back and forth between the two worlds—the real and the mathematical—as the need arises. For a particular problem the problem solver might move directly along arrow Y_1 from the real world to

tion of one suggested for students in grade 2 (Alberta Education 1983, p. 54). This problem was used by Croft (1987), a teacher who conducted individual interviews with several of her grade 1 students.

Croft noticed that different children had distinct levels of understanding that corresponded to the number of different mathematical concepts and processes they used in solving the problem. All the children began by sorting the coins by value and repeatedly using one-to-one correspondence; they placed the four dimes into four piles, then four of the nickels, then four more nickels, then the pennies. But some students got stuck when they reached the situation shown in figure 3.4. Although they were satisfied that the coins in each of the four piles matched, they did not know what to do with the nickel and three pennies that were left over. The children at the lowest level of understanding never got beyond this impasse, despite being encouraged to "try a different way" and to "share out all the coins." One child suggested that he could solve the problem if only he could take the nickel to the store and exchange it for five pennies. Other children used the making-change idea and recognized that by "undoing" part of the sharing, they could remove the three pennies from one pile and replace them with the "extra" nickel so that six pennies could then be put in the other three piles. This solution represents the next level of understanding. When the teacher asked, "Can you find another way? Could the piles have different coins in them but the same amount of money?" some children rearranged the coins but found no new solutions. However, the students at the highest level of understanding noticed that the value of the coins in each pile was twenty-five cents and used this fact to find several different ways of making change, such as those shown in figure 3.5.

These differences in students' performance indicate the variety of mathematical operations inherent in the problem, including sorting, finding a one-to-one correspondence, iterating, exchanging sets of equal value, and counting the value of a collection of coins and using the value (rather than the coins themselves) to find other collections of the same value. The pupils'

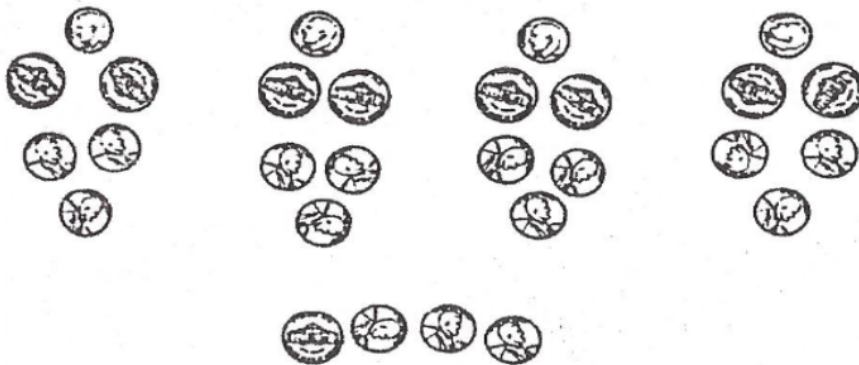


Fig. 3.4. An impasse on the way to solving the coin problem

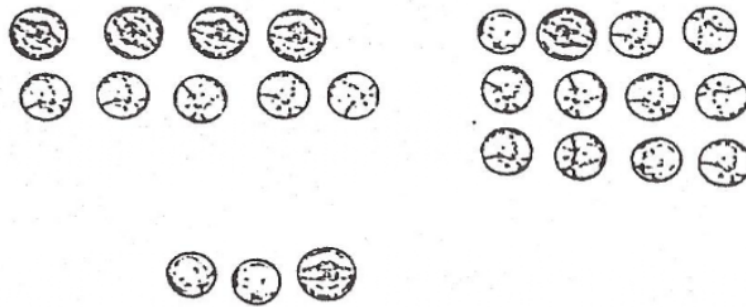


Fig. 3.5. Some different ways to make twenty-five cents

ability to recognize and use these ideas gives a measure of their understanding. It is interesting to note that some children's understanding seemed to deepen and grow as they worked on the problem; their progress with the problem came in stages, by discovery, rather than all at once. This suggests that these students were learning *via* problem solving, even though the teacher's purpose was to assess their understanding rather than to teach them via problem solving.

DEVELOPING UNDERSTANDING VIA PROBLEM SOLVING

We believe that instead of making *problem solving* the focus of mathematics instruction, teachers, textbook authors, curriculum developers, and evaluators should make *understanding* their focus and their goal. By doing so they will shift from the narrow view that mathematics is simply a tool for solving problems to the broader conception that mathematics is a way of thinking about and organizing one's experiences. As a consequence, problem solving will not be de-emphasized, but the role of problem solving in the curriculum will change from being an activity students engage in after they have acquired certain concepts and skills to being both a means for acquiring new mathematical knowledge and a process for applying what has been learned previously. Fundamental to the view that understanding should be a primary goal of instruction is the belief that children's learning of mathematics is richest when it is self-generated rather than when it is imposed by a teacher or textbook. A primary advantage of self-generated knowledge is that it is tied to what the learner already knows. Furthermore, when children construct new mathematical knowledge for themselves, they learn not only concepts, facts, skills, and so on, but also how to manage and regulate the application of this new knowledge. That is, they are in charge of this knowledge (and of their learning in general), thereby making it more useful to them in solving problems and in learning new concepts and skills. A benefit of having acquired mathematical knowledge in this way is that problem-solving efforts are less susceptible to error. We believe that teach-

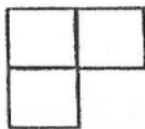
ing via problem solving and teaching for understanding are not only compatible but in fact mutually beneficial.

Problem Solving Enhances Understanding

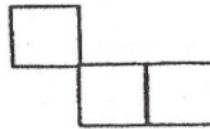
Beatriz D'Ambrosio has suggested the following challenge as an illustration of the fact that solving a problem can deepen a student's understanding of a topic of mathematics.

On centimeter graph paper outline all the shapes that have an area of 14 square cm and a perimeter of 24 cm. For each shape you draw, at least one side of each square must share a side with another square. Here's an example:

Allowed



Not allowed



It is assumed, of course, that students given this problem would already have a basic understanding of the concepts of area and perimeter for rectangular shapes. The intention is not simply to allow the students an opportunity to apply their knowledge of these two concepts. Rather, it is to enhance their understanding of the relationships between area and perimeter. The solution to this problem requires that students make many decisions, among them how to keep track in a systematic way of the shapes that have been made so that all possibilities will be found and none will be duplicated. Such decisions and the associated skills needed to carry them out are an important part of learning how to solve problems successfully and efficiently. But learning what decisions to make and when to make them is not the only benefit of this task. In addition, as shapes are modified to fit the conditions of the problem, the learner is exposed to relationships between area and perimeter that, if noticed, can facilitate a richer understanding of both concepts. Thus, through investigation and exploration, students not only learn some useful problem-solving skills but also deepen their understanding of two important measurement concepts.

Understanding Aids Problem Solving

Of course, success in solving a problem depends on the student's having a good understanding of the information in it. However, the value of understanding in successful problem solving goes far beyond this. In particular, when understanding is viewed in the way we have discussed, it aids problem solving in at least four distinct ways.

1. *Understanding increases the richness of the types of representations that the problem solver can construct.* During problem solving it is necessary for the problem solver to internalize the information in a problem. That is, the problem solver must develop a representation of the information. The more accurately the representation depicts the information and links pieces of information together, the more likely it is that the problem will be solved correctly.

2. *Understanding assists the problem solver in monitoring the selection and execution of procedures (e.g., strategies, algorithms).* Successful problem solving requires the ability to monitor the selection and subsequent execution of procedures, and the ability to evaluate the extent to which local actions (e.g., performing computations) conform to goals, and the ability to make various trade-off decisions (e.g., deciding that an estimate will give a "close enough" answer). The problem solver who understands the relationships among the conditions and variables in a problem and who can place the problem in a meaningful context is well equipped to anticipate the consequences of various decisions and actions and to evaluate the progress being made toward a solution.

3. *Understanding aids the problem solver in judging the reasonableness of results.* The ability to create a meaningful and appropriate internal representation of the information in a problem enhances the problem solver's ability to determine whether the answer makes sense.

4. *Understanding promotes the transfer of knowledge to related problems and its generalizability to other situations.* Brownell (1947), among others, has pointed out that a solution to a problem that is meaningful (i.e., well understood) transfers readily to problems that are similar in structure even if they are different in context. That is, since understanding involves the ability to apply a particular concept, skill, or procedure to unfamiliar situations, an individual who has a good understanding of certain mathematical ideas and techniques is likely to be able to apply that learning to contexts that might be very different from the contexts in which the mathematics was originally learned.

CONCLUSION

We believe that there can be a mutually supportive relationship between emphasizing problem solving and emphasizing understanding in mathematics instruction. When teachers teach *via* problem solving, as well as *about* it and *for* it, they provide their students with a powerful and important means of developing their own understanding. As students' understanding of mathematics becomes deeper and richer, their ability to use mathematics to solve problems increases.

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